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Curvatures of the Factorable Hypersurface

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

The curvatures $\mathfrak{C}_{\mathrm{i}=1,2,3}$ of a factorable hypersurface are introduced in the four-dimensional Euclidean space. It is also given some relations on $\mathfrak{C}_{\mathrm{i}}$ of the factorable hypersurface.

Keywords: Four-space; factorable hypersurface; fourth fundamental form.

1 Introduction

Surfaces and hypersurfaces have been studied by mathematicians for centuries. It can be seen some papers about factorable surfaces and factorable hypersurfaces in the literature such as [1–25].

A factorable hypersurface in $\mathbb{E}^4$ can be parametrized by

$$\mathbf{x}(u, v, w) = (u, v, w, uvw), \tag{1.1}$$

where $u, v, w \in I \subset \mathbb{R}$.

In this paper, the fourth fundamental form of the factorable hypersurface is obtained in the four-dimensional Euclidean space $\mathbb{E}^4$. Some notions of four-dimensional Euclidean geometry are shown. Moreover, the curvatures $\mathfrak{C}_{i=1,2,3}$ of the factorable hypersurface are obtained.

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2 Preliminaries

Characteristic polynomial of the shape operator **S** is obtained by as follows

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \tag{2.1}$$

where I_n denotes the identity matrix of order n in $\mathbb{E}^{n+1}$. Then, curvature formulas are defined by as follows

$$\binom{n}{i} \mathfrak{C}_i = s_i,$$

where $\binom{n}{0} \mathfrak{C}_0 = s_0 = 1$ by definition. Therefore, k-th fundamental form of hypersurface M^n is given by

$$\mathrm{I}(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y \rangle.$$

Hence

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \, \mathrm{I}(\mathbf{S}^{k-1}(X), Y) = 0 \tag{2.2}$$

is hold.

A vector $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$ with its transpose are considered as identify in this work.

Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in $\mathbb{E}^4$. The inner product of vectors $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$ is given by as follows:

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{4} x_i y_i.$$

Vector product $\vec{x} \times \vec{y} \times \vec{z}$ of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{E}^4$ is defined by as follows:

$$\vec{x} \times \vec{y} \times \vec{z} = \det \begin{pmatrix} e_1 \, e_2 \, e_3 \, e_4 \\ x_1 x_2 x_3 x_4 \\ y_1 y_2 y_3 y_4 \\ z_1 \, z_2 \, z_3 \, z_4 \end{pmatrix}.$$

The Gauss map of a hypersurface **M** is given by

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|},$$

where $\mathbf{M}_u = d\mathbf{M}/du$. For a hypersurface **M** in $\mathbb{E}^4$, following fundamental form matrices are holds:

$$\mathrm{I} = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix},$$

$$\mathrm{II} = \det\begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix},$$

$$\mathrm{III} = \begin{pmatrix} X & Y & O \\ Y & Z & R \\ O & R & S \end{pmatrix},$$

where the coefficients are given by

$$E = \langle \mathbf{M}_u, \mathbf{M}_u \rangle,\ \ F = \langle \mathbf{M}_u, \mathbf{M}_v \rangle,\ \ G = \langle \mathbf{M}_v, \mathbf{M}_v \rangle,\ \ A = \langle \mathbf{M}_u, \mathbf{M}_w \rangle,\ \ B = \langle \mathbf{M}_v, \mathbf{M}_w \rangle,\ \ C = \langle \mathbf{M}_w, \mathbf{M}_w \rangle,$$

$$L = \langle \mathbf{M}_{uu}, e \rangle,\ \ M = \langle \mathbf{M}_{uv}, e \rangle,\ \ N = \langle \mathbf{M}_{vv}, e \rangle,\ \ P = \langle \mathbf{M}_{uw}, e \rangle,\ \ T = \langle \mathbf{M}_{vw}, e \rangle,\ \ V = \langle \mathbf{M}_{ww}, e \rangle,$$

$$X = \langle e_u, e_u \rangle,\ \ Y = \langle e_u, e_v \rangle,\ \ Z = \langle e_v, e_v \rangle,\ \ O = \langle e_u, e_w \rangle,\ \ R = \langle e_v, e_w \rangle,\ \ S = \langle e_w, e_w \rangle.$$

3 Curvatures

Next, the curvatures of a hypersurface $\mathbf{M}(u, v, w)$ will be obtained in $\mathbb{E}^4$. Using characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, the curvature formulas are computed: $\mathfrak{C}_0 = 1$ (by definition),

$$\binom{3}{1}\mathfrak{C}_1 = -\frac{b}{a},\ \ \binom{3}{2}\mathfrak{C}_2 = \frac{c}{a},\ \ \binom{3}{3}\mathfrak{C}_3 = -\frac{d}{a}.$$

Then, the following curvature formulas are hold:

3.1 Theorem

Any hypersurface M^3 in $\mathbb{E}^4$ has following curvature formulas, $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{(EN + GL - 2FM)C + (EG - F^2)V - LB^2 - NA^2 - 2(APG - BPF - ATF + BTE - ABM)}{3[(EG - F^2)C - EB^2 + 2FAB - GA^2]}, \tag{3.1}$$

$$\mathfrak{C}_2 = \frac{(EN + GL - 2FM)V + (LN - M^2)C - ET^2 - GP^2 - 2(APN - BPM - ATM + BTL - PTF)}{3[(EG - F^2)C - EB^2 + 2FAB - GA^2]}, \tag{3.2}$$

$$\mathfrak{C}_3 = \frac{(LN - M^2)V - LT^2 + 2MPT - NP^2}{(EG - F^2)C - EB^2 + 2FAB - GA^2}. \tag{3.3}$$

Proof. Solving $\det(\mathbf{S} - \lambda I_3) = 0$ with some calculations, the coefficients of polynomial $P_{\mathbf{S}}(\lambda)$ are found.

3.2 Theorem

For *any hypersurface M^3 in $\mathbb{E}^4$, curvatures are related by following formula*

$$\mathfrak{C}_0 IV - 3\mathfrak{C}_1 III + 3\mathfrak{C}_2 \mathrm{II} - \mathfrak{C}_3 I = 0. \tag{3.4}$$

4 Curvatures of factorable hypersurface

The curvatures of factorable hypersurface (1.1) will be computed in this section.

With the first differentials of (1.1) depends on u, v, w, the Gauss map of (1.1) is given by

$$e = \frac{1}{(\det \mathrm{I})^{1/2}} \begin{pmatrix} v\,w \\ u\,w \\ u\,v \\ -1 \end{pmatrix}. \tag{4.1}$$

$\det \mathrm{I} = u^2v^2 + u^2w^2 + v^2w^2 + 1$. The first and the second fundamental form matrices of (1.1) are found by as follows, respectively,

$$\mathrm{I} = \begin{pmatrix} v^2w^2+1 & uvw^2 & uv^2w \\ uvw^2 & u^2w^2+1 & u^2vw \\ uv^2w & u^2vw & u^2v^2+1 \end{pmatrix},$$

$$\mathrm{II} = \begin{pmatrix} 0 & -\frac{w}{(\det \mathrm{I})^{1/2}} & -\frac{v}{(\det \mathrm{I})^{1/2}} \\ -\frac{w}{(\det \mathrm{I})^{1/2}} & 0 & -\frac{u}{(\det \mathrm{I})^{1/2}} \\ -\frac{v}{(\det \mathrm{I})^{1/2}} & -\frac{u}{(\det \mathrm{I})^{1/2}} & 0 \end{pmatrix}.$$

Computing matrix $I^{-1} \cdot II$, shape operator matrix of the factorable hypersurface (1.1) can be seen as follows

$$\mathbf{S} = \begin{pmatrix} \frac{uvw(v^2+w^2)}{(\det \mathrm{I})^{3/2}} & -\frac{w(u^2w^2+1)}{(\det \mathrm{I})^{3/2}} & -\frac{v(u^2v^2+1)}{(\det \mathrm{I})^{3/2}} \\ -\frac{w(v^2w^2+1)}{(\det \mathrm{I})^{3/2}} & \frac{uvw(u^2+w^2)}{(\det \mathrm{I})^{3/2}} & -\frac{u(u^2v^2+1)}{(\det \mathrm{I})^{3/2}} \\ -\frac{v(v^2w^2+1)}{(\det \mathrm{I})^{3/2}} & -\frac{u(u^2w^2+1)}{(\det \mathrm{I})^{3/2}} & \frac{uvw(u^2+v^2)}{(\det \mathrm{I})^{3/2}} \end{pmatrix}.$$

4.1 Theorem

Factorable hypersurface (1.1) *in* $\mathbb{E}^4$ *has the following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{2uvw(u^2+v^2+w^2)}{3(u^2v^2+u^2w^2+v^2w^2+1)^{3/2}},$$

$$\mathfrak{C}_2 = \frac{3u^2v^2w^2 - (u^2+v^2+w^2)}{3(u^2v^2+u^2w^2+v^2w^2+1)^2},$$

$$\mathfrak{C}_3 = -\frac{2uvw}{(u^2v^2+u^2w^2+v^2w^2+1)^{5/2}}.$$

Proof. Computing (3.1), (3.2), and (3.3) of (1.1), the curvatures is obtained.

4.2 Corollary

Factorable hypersurface (1.1) *in* $\mathbb{E}^4$ *has the following relations*

$$\frac{(\mathfrak{C}_1)^2\mathfrak{C}_2}{(\mathfrak{C}_3)^2} = \frac{(3p^2-q)q^2}{9}.$$

Where

$$p = uvw,\ \ q = u^2 + v^2 + w^2.$$

Proof. Using Theorem 4.1, it is seen clearly.

4.3 Corollary

The factorable *hypersurface* (1.1) *depends on* $\mathfrak{C}_1$ *in* $\mathbb{E}^4$ *can be written as follows*

$$\mathbf{x}(u,v,w) = \left(u, v, w, \frac{3\mathfrak{C}_1(\det\mathrm{I})^{3/2}}{q}\right).$$

4.4 Corollary

The factorable hypersurface (1.1) *depends on* $\mathfrak{C}_2$ *in* $\mathbb{E}^4$ *can be written as follows*

$$\mathbf{x}(u,v,w) = \left(u, v, w, \pm\left(\frac{3\mathfrak{C}_2(\det\mathrm{I})^2 + q}{3}\right)^{1/2}\right).$$

4.5 Corollary

The factorable hypersurface (1.1) *depends on* $\mathfrak{C}_3$ *in* $\mathbb{E}^4$ *can be written as follows*

$$\mathbf{x}(u,v,w) = \left(u, v, w, -\frac{\mathfrak{C}_3(\det\mathrm{I})^{5/2}}{2}\right).$$

5 Conclusion

Factorable hyper-surfaces have been studied by lots of authors for a long time. Results of the factorable hypersurface (1.1) are expanded by using its curvatures in $\mathbb{E}^4$. In addition, factorable hypersurface (1.1) are given by its curvatures $\mathfrak{C}_1, \mathfrak{C}_2$, and $\mathfrak{C}_3$ of $\mathbb{E}^4$ in this work.

Competing Interests

Author has declared that no competing interests exist.

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