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A Family of Graceful Diameter Six Trees Generated by Component Moving Techniques

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Authors' contributions

This work was carried out in collaboration between both authors. Author DM identified the problem, devised the solution methodology based on component moving transformation, and wrote the final draft of the article. Author ACP scripted the first draft of the article and gave all the examples. Both authors read and approved the final manuscript.

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Original Research Article

Abstract

Aims/ Objectives: To identify some new classes of graceful diameter six trees using component moving transformation techniques.
Study Design: Literature Survey to our findings.
Place and Duration of Study: Department of Mathematics,C.V. Raman College Of Engineering,Bhubaneswar, India, between June 2014 and September 2016.
Methodology: Component Moving Transformation.
Results: Here a diameter six tree is denoted by $(a_0;\ a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots,\ b_n;\ c_1, c_2, \ldots, c_r)$ with a_0 as the center of the tree, $a_i,\ i = 1, 2, \ldots, m$, $b_j,\ j = 1, 2, \ldots, n$, and $c_k,\ k = 1, 2, \ldots, r$ are the vertices of the tree adjacent to a_0; each a_i is the center of some diameter four tree, each b_j is the center of some star, and each c_k is some pendant vertex. This article gives graceful labelings

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to a family of diameter six trees $(a_0;\ a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots,\ b_n;\ c_1, c_2, \ldots, c_r)$ with diameter four trees incident on a_is possess an odd number of branches comprising of six different combinations of odd, even, and pendant branches. Here a star is called an odd branch if its center has an even degree, an even branch if its center has an odd degree, and a pendant branch if its center has degree one.
Conclusions: Our article finds many new graceful diameter six trees by component moving techniques. However, the problem that all diameter six trees are graceful is still open and we conclude that one can not give graceful labelings to all diameter six trees by component moving techniques.

Keywords: Graceful labeling; diameter six tree; component moving transformation; transfers of the first and second types; BD8TF.

AMS classification: 05C78.

1 Introduction

By a graph labelling we mean an assignment of integers to the vertices or edges or both, subject to certain conditions. The concept of graph labelling originated in in the 1960s while attempting to resolve the problems involving decomposition of graphs into smaller graphs. In last five and half decades many new graph labelling techniques have evolved and more than 1500 research articles are available so far in this area. Labelled graphs have also been implemented in many problems in applied sciences and Engineering such as - network addressing, X - ray crystallography, coding theory, rulers, radar and missile guidance, constrained satisfactory problems, radio antenna problems [1], [2]. In this article we have undertaken a study on a very fundamental and widely used graph labelling, namely, *graceful labelling.* Graceful labeling was introduced by Ringel [3], Kotzig [4], and Rosa [5] and it is defined as follows.

Definition 1.1. [6], [5] A graph G with q edges is said to be *graceful* if there is an injection f from the vertices of G to the set $\{0, 1, 2, 3, \ldots, q\}$ such that set of absolute values of difference of the vertex labels of all the edges of G is the set $\{1, 2, 3, \ldots, q\}$.

The concept of graceful labeling came into existence while trying to resolve an conjecture due to Ringel [3] which states that "K_{2n+1} decomposes intro $2n + 1$ isomorphic copies of a tree with n edges." Rosa [5] proved that Ringel's conjecture holds good if the tree is graceful. Rosa [5] also conjectured that all trees are graceful, which is popularly known as graceful tree conjecture. Despite of many efforts in past five decades the graceful tree conjecture remains unresolved so far.

From the available literature and most up to date surveys on graph labeling problems (Edwards and Howard [7], Gallian [6], Hrnciar and Havier[8], Robeva [9], Rosa [5]) it has been established that all trees up to diameter five are graceful. Here we give graceful labelings to certain classes of diameter six trees. Here we first give a representation of a diameter six tree as given below.

Definition 1.2. [10], [11], [12] A *diameter six tree* can be represented as $(a_0;\ a_1, a_2, \ldots, a_m;$ $b_1, b_2, \ldots, b_n;\ c_1, c_2, \ldots, c_r)$, where a_0 is the center of the tree; a_i; b_j, and c_k are the vertices of the tree adjacent to a_0 such that each a_i is the center of some diameter four tree, each b_j is the center of some star, and each c_k is some pendant vertex. It is readily observed that for a diameter six tree with the above representation there are at least two neigbours of a_0 which are the centers of diameter four trees. The notation D_6 shall henceforth represent a diameter six tree.

A combination of branches incident on any a_i, $0 \leq i \leq m$ is a triple (x, y, z), where x, y, and z denote the number of odd, even, and pendant branches, respectively, incident on a_i. Here the symbols e and o denote a non-zero even number and an odd number, respectively. For example: $(o, 0, e)$ means an odd number of odd branches, no even branch, and an even number of pendant branches. If in a triple e or o appears more than once then it does not mean that the corresponding branches are equal in number, for example (o, o, e) does not mean that the number of odd branches is equal to the number of even branches.

As far as diameter six trees are concerned *banana trees* are known to be graceful [13], [14], [7], [6], [15], [16], [9], [17], [18], [19]. Chen et. al. [14] defined a *banana tree* as a tree obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). Chen et. al. [14] conjectured that banana trees are graceful. Bhatt Nayak and Deshmukh [13], Murugan and Arumurugan [16] and Vilfred [19] gave graceful labelings to different classes of banana trees.

Sethuraman and Jesintha [15], [17], [18] proved that all banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful. Mishra and Panda [10], [11], and [12] developed some classes of graceful diameter six trees.

Applying the techniques of Hrnciar and Havier [8], Mishra and Panda [20], and Mishra and Panigrahi [21], [22] here we give graceful labelings to some new classes of diameter six trees $(a_0; a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots,\ b_n;\ c_1, c_2, \ldots, c_r)$ with each a_i, $i = 1, 2, \ldots, m_1, m_1 \leq m$, is attached to $(o, 0, 0)$ and the branches incident on a_i, $m_1 + 1 \leq m$, satisfies one of the following conditions.

1. Each a_i, $m_1 + 1 \leq m_2$ is attached to $(o, 0, e)$ (or $(o, e, 0)$). Each a_i, $m_2 + 1 \leq m_3$ is attached to (o, e, e) or $(e, o, 0)$. Each a_i, $m_3 + 1 \leq m_4$ is attached to (e, o, e) or $(e, o, 0)$. Each a_i, $m_4 + 1 \leq m_5$ is attached to $(0, o, e)$ or (e, e, o) and each of the remaining $a_i's$ is attached to $(e, 0, o)$ or $(0, e, o)$.
2.Each a_i, $m_1 + 1 \leq m_2$ is attached to $(o, 0, e)$ (or $(o, e, 0)$). Each a_i, $m_2 + 1 \leq m_3$ is attached to (o, e, e). Each a_i, $m_3 + 1 \leq m_4$ is attached to (o, o, o) and each of the remaining $a_i s$ is attached to $(e, 0, o)$.
3.Each a_i, $m_1 + 1 \leq m_2$ is attached to $(o, 0, e)$. Each a_i, $m_2 + 1 \leq m_3$ is attached to (o, o, o). Each a_i, $m_3 + 1 \leq m_4$ is attached to (e, e, o) and each of the remaining $a_i s$ is attached to $(e, 0, o)$ or $(0, e, o)$.

2 Preliminaries

Definition 2.1. [8], [20], [21], [22] For an edge $e = \{u, v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a component incident on the vertex v. If a and b are vertices of a tree T, $u(T)$ is a component incident on a, and $b \notin u(T)$ then deleting the edge $\{a, u\}$ from T and making b and u adjacent is termed as *the component $u(T)$ has been transferred or moved from a to b.* In this paper by the label of the component "$u(T)$" we mean the label of the vertex u. Let T be a tree and a and b be two vertices of T. By $a \rightarrow b$ *transfer* we mean that some components from a have been moved to b. If we consider successive transfers $a_1 \rightarrow a_2$, $a_2 \rightarrow a_3$, $a_3 \rightarrow a_4$, $\ldots$ we simply write $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4 \ldots$ transfer. In the transfer $a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_{n-1} \rightarrow a_n$, each vertex a_i, $i = 1, 2, \ldots,\ n - 1$ is called a vertex of transfer. Let T be a labelled tree with a labeling f. We consider the vertices of T whose labels form the sequence $(a, b, a-1, b+1, a-2, b+2)$ (respectively, $(a, b, a+1, b-1, a+2, b-2)$). Let a be adjacent to some vertices having labels different from the above labels. The $a \longrightarrow b$ transfer is called *a transfer of the first type* if the labels of the transferred components constitute a set of consecutive integers.

The $a \longrightarrow b$ transfer is called *a transfer of the second type* if the labels of the transferred components can be divided into two segments, where each segment is a set of consecutive integers. A sequence of eight transfers of the first type $a \to b \to a-1 \to b+1 \to a \to b \to a-1 \to b+1 \to a-2$ (respectively, $a \to b \to a+1 \to b-1 \to a \to b \to a+1 \to b-1 \to a+2$), is called a *backward double 8 transfer of the first type* or $BD8TF$ a to $a-2$ (respectively, a to $a+2$).

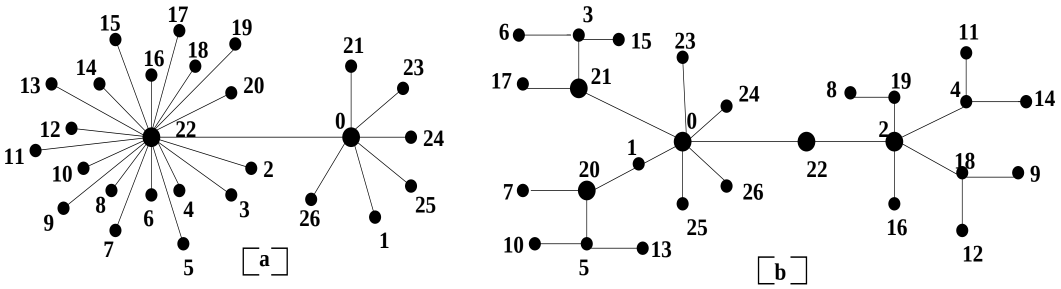

Fig. 1. The graceful tree in (b) is obtained from the graceful tree in (a) by carrying out a sequence of transfers consisting of $22 \to 1 \to 21$ transfers of the first type, followed by the BD8TF 21 to 19, followed by $19 \to 4$ transfer of the first type, and finally $4 \to 18 \to 5$ transfers of the second type, respectively

Theorem 2.1. [20], [21], [22] In a graceful labeling f of a graceful tree T, let a and b be the labels of two vertices. Let a be attached to a set A of vertices (or components) having labels $n, n+1, n+2, \ldots, n+p$ (different from the above vertex labels), which satisfy $(n+1+i)+(n+p-i) = a+b,\ i \geq 0$ (respectively, $(n+i)+(n+p-1-i) = a+b,\ i \geq 0$). Then the following hold.

(a) By making a transfer $a \to b$ of first type we can keep an odd number of components at a from the set A and move the rest to b, and the resultant tree thus formed will be graceful.

(b) If A contains an even number of elements, then by making a sequence of transfers of the second type $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \ldots$ (respectively, $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \ldots$), an even number of elements from A can be kept at each vertex of the transfer, and the resultant tree thus formed is graceful.

(c) By a BD8TF a to $b+1$ (respectively, $b-1$), we can keep an even number of elements from A at $a, b, a-1$, and $b+1$ (respectively, $a, b, a+1$, and $b-1$), and move the rest to $a-2$ (respectively, $a+2$). The resultant tree formed in each of the above cases is graceful.

(d) Consider the transfer $R' : a \to b \to a-1 \to b+1 \to \ldots \to \ldots$ (respectively, $a \to b \to a+1 \to b-1 \to \ldots \to \ldots$), such that R' is partitioned as $R' : T'_1 \to T'_2$, where T'_1 is sequence of transfers consisting of the transfers of the first type and BD8TF and T'_2 is a sequence of transfer of the second type. The tree T^{**} obtained from T by making the transfer R' is graceful.

Lemma 2.2. [8] If g is a graceful labeling of a tree T with n edges then the labeling g_n defined as $g_n(x) = n - g(x)$, for all $x \in V(T)$, called the *inverse transformation* of g is also a graceful labeling of T.

3 Results

Theorem 3.1. If degrees of a_i and b_j are even, for $i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n$, and the centers a_i, $i = 1, 2, \dots, m$, of diameter four trees are attached to combinations as shown in Table 1 then D_6 given by the following are graceful.

(a): $D_6 = \{a_0; a_1, a_2, \dots, a_m; b_1, b_2, \dots, b_n; c_1, c_2, \dots, c_r\}$.
(b): $D_6 = \{a_0; a_1, a_2, \dots, a_m; b_1, b_2, \dots, b_n\}$.
(c): $D_6 = \{a_0; a_1, a_2, \dots, a_m; c_1, c_2, \dots, c_r\}$ with m odd.
(d): $D_6 = \{a_0; a_1, a_2, \dots, a_m\}$ with m odd.

Table 1. Diameter Six Trees of Theorem 3.1.

Cases ↓	a_i, $1 \le i \le m_1$	a_i, $m_1 + 1 \le i \le m_2$	a_i, $m_2 + 1 \le i \le m_3$	a_i, $m_3 + 1 \le i \le m_4$	a_i, $m_4 + 1 \le i \le m_5$	a_i, $m_5+1 \le i \le m_6 = m$
(a)	$(o, 0, 0)$	$(o, e, 0)$	$(e, o, 0)$	(e, o, e)	$(0, o, e)$	$(0, e, o)$
(b)	same as (a)	same as (a)	same as (a)	same as (a)	(e, e, o)	same as (a)
(c)	same as (a)	same as (a)	(o, e, e)	same as (a)	same as (b)	same as (a)
(d)	same as (a)	same as (a)	same as (a)	same as (a)	same as (b)	$(e, 0, o)$
(e)	same as (a)	same as (a)	same as (c)	same as (a)	same as (b)	same as (d)
(f)	same as (a)	$(o, 0, e)$	same as (c)	same as (a)	same as (b)	same as (d)
(g)	same as (a)	same as (f)	same as (c)	same as (a)	same as (a)	same as (a)
(h)	same as (a)	same as (f)	same as (c)	same as (a)	same as (b)	same as (a)
(i)	same as (a)	same as (a)	same as (c)	same as (a)	same as (a)	same as (a)

Proof (a): Case - I Let $m + n$ be odd. Let $|E(D_6)| = q$ and $deg(a_0) = m + n = 2k + 1$. Proceed as per the following steps.

1. Remove the pendant vertices adjacent to a_0 and represent the new graceful tree by $D_6^{(1)}$. Consider the graceful tree G as represented in Fig. 2.
2. Define integers $\alpha_i^{(j)}$, for $i = 1, 2, \dots, m$, $j = 1, 2, 3, 4, 5$, as per the following.

For $1 \le i \le m_1$**:** $2\alpha_i^{(1)} + 1 = o_i = deg(a_i) - 1$.

For $m_1+1 \le i \le m_2$**:** For the cases (a), (b), (c), (d), (e), and (i): $2\alpha_i^{(1)}+1 = o_i$, $2[\alpha_i^{(2)}+\alpha_i^{(3)}+1] = e_i$. For the cases (f), (g), and (h): $2\alpha_i^{(1)} + 1 = o_i$, $2[\alpha_i^{(4)} + \alpha_i^{(5)} + 1] = p_i$.

For $m_2 + 1 \le i \le m_3$**:** For the cases (a), (b) and (d): $2[\alpha_i^{(1)} + \alpha_i^{(2)} + 1] = o_i$, $2\alpha_i^{(3)} + 1 = e_i$. For the cases (c), (e), (f), (g), (h), and (i): $2\alpha_i^{(1)} + 1 = o_i$, $2[\alpha_i^{(2)} + \alpha_i^{(3)} + 1] = e_i$, and $2[\alpha_i^{(4)} + \alpha_i^{(5)} + 1] = p_i$.

For $m_3 + 1 \le i \le m_4$**:** $2[\alpha_i^{(1)} + \alpha_i^{(2)} + 1] = o_i$, $2\alpha_i^{(3)} + 1 = e_i$, and $2[\alpha_i^{(4)} + \alpha_i^{(5)} + 1] = p_i$.

For $m_4 + 1 \le i \le m_5$**:** For the cases (a), (g), and (i): $2\alpha_i^{(3)} + 1 = e_i$ and $2[\alpha_i^{(4)} + \alpha_i^{(5)} + 1] = p_i$. For the cases (b), (c), (d), (e), (f), and (h): $2[\alpha_i^{(1)} + \alpha_i^{(2)} + 1] = o_i$, $2[\alpha_i^{(3)} + \alpha_i^{(4)} + 1] = e_i$, and $2\alpha_i^{(5)} + 1 = p_i$.

For $m_5 + 1 \le i \le m_6 = m$**:** For the cases (a), (b), (c), (g), (h), and (i): $2[\alpha_i^{(3)} + \alpha_i^{(4)} + 1] = e_i$, and $2\alpha_i^{(5)}+1 = p_i$. For the cases (d), (e), and (f): $2[\alpha_i^{(1)}+\alpha_i^{(2)}+1] = o_i$ and $2\alpha_i^{(5)}+1 = p_i$.

3. Let $A=\{k+1,k+2,\dots,q-k-r-1\}$. Observe that $(k+i)+(q-r-k-i) = q-r$. Assign the labels to $a_i,\ 1\le i\le m$ and $b_j,\ 1\le j\le n$ as follows.

$x_i = \begin{cases} q-r-\frac{i-1}{2} & \text{if } i \text{ is odd} \\ \frac{i}{2} & \text{if } i \text{ is even} \end{cases}$ with x_i is the label of a_i, for $i = 1,2,\dots,m$ and x_{m+j} is the label of b_j, for $j = 1,2,\ \dots,\ n$.

4. Define an integer t as $t = a_{m_4+1}$ for the cases (a) and (g), $t = a_{m_5+1}$ for the cases (b) and (c), and $t = b_1$ for the cases (d), (e), and (f). Observe that the transfer $T_1:\ a_1\to a_2\to a_3\to a_4\to a_5\to a_6\to\dots\to t$ and the set A satisfy the hypothesis of Theorem 2.1. Carry out the transfer T_1 consisting of the successive transfers of the first type and keep $2\alpha_i^{(1)}+1$ elements of A at the vertices a_i of T_1. Let A_1 be the set of vertices of A that have come to the vertex t.

5. Define an integer t_1 as $t_1 = a_{m_4}$ for the cases (a) and (g), $t_1 = a_{m_5}$ for the cases (b) and (c), and $t_1 = a_m$ for the cases (d), (e), and (f). Carry out the transfer $t\longrightarrow t_1$ of the first type and bring back all the elements of A_1 to t_1. Obviously, the new tree thus formed, say G_2, is graceful.

6. Define the transfer T_2 as follows. $T_2: a_{m_4}\to a_{m_4-1}\to\dots\to a_{m_1+1}\to a_{m_1}$ for the cases (a) and (i); $T_2: a_{m_5}\to a_{m_5-1}\to\dots\to a_{m_1+1}\to a_{m_1}$ for the cases (b) and (c); $T_2: a_m\to a_{m-1}\to\dots\to a_{m_1+1}\to a_{m_1}$ for the cases (d) and (e); $T_2: a_m\to a_{m-1}\to\dots a_{m_2+1}\to a_{m_2}$ for the case (f); $T_2: a_{m_4+1}\to a_{m_4}\to\dots\to a_{m_2+1}\to a_{m_2}$ for the cases (g); $T_2: a_{m_5}\to a_{m_5-1}\to\dots\to a_{m_2+1}\to a_{m_2}$ for the case (h). Observe that the set A_1 and the labels of the vertices of T_2 satisfy the hypothesis of Theorem 2.1. Carry out the transfer T_2 consisting of successive transfers of the first type, keeping $2\alpha_i^{(2)}+1$ elements of A_1 at the vertices a_i of T_2. By Theorem 2.1 the new tree, say G_3, thus formed is graceful. Let A_2 be the set of vertices of A_1 which have been transferred to the last vertex of T_2.

7. Execute the transfer $a_{m_1}\to a_{m_1+1}$ (for the cases (a), (b), (c), (d), (e), and (i)) or $a_{m_2}\to a_{m_2+1}$ (for the cases (f), (g), and (h)) and bring back all elements of A_2 to a_{m_1} or a_{m_2}.

8. Define the transfer T_3 as follows. $T_3:\ a_{m_1+1}\to a_{m_1+2}\to a_{m_1+3}\to\dots\to a_m\to b_1$ for the cases (a), (b), (c), and (i); $T_3:\ a_{m_1+1}\to a_{m_1+2}\to a_{m_1+3}\to\dots\to a_{m_5}\to a_{m_5+1}$ for the cases (d) and (e); $T_3:\ a_{m_2+1}\to a_{m_2+2}\to a_{m_2+3}\to\dots\to a_{m_5}\to a_{m_5+1}$ for the case (f); $T_3:\ a_{m_2+1}\to a_{m_2+2}\to a_{m_2+3}\to\dots\to a_m\to b_1$ for the cases (g) and (h). Observe that the transfer T_3 and the set A_2 satisfy the hypothesis of Theorem 2.1. Execute the transfer T_3 consisting of the successive transfers of the first type and keep $2\alpha_i^{(3)}+1$ elements of A_2 at the vertices a_i of T_3. Let A_3 is the set of vertices of A_2 that have come to the vertex b_1 or a_{m_5+1} as the case may be.

9. Execute the transfer $b_1\longrightarrow a_m$ (or $a_{m_5+1}\to a_{m_5}$) of the first type and bring back all the elements of A_3 to a_m (or a_{m_5}). Obviously, the new tree thus formed, say G_5, is graceful.

10. Define the transfer T_4 as follows. $T_4: a_m\to a_{m-1}\to a_{m-2}\to a_{m-3}\to\dots\to a_{m_3+1}\to a_{m_3+1}\to a_{m_3}$ for the cases (a), (b), and (i); $T_4: a_m\to a_{m-1}\to a_{m-2}\to a_{m-3}\to\dots\to a_{m_2+1}\to a_{m_2+1}\to a_{m_2}$ for the case (c); $T_4: a_{m_5}\to a_{m_5-1}\to a_{m_5-2}\to a_{m_5-3}\to\dots\to a_{m_3+1}\to a_{m_3+1}\to a_{m_3}$ for the cases (d); $T_4: a_{m_5}\to a_{m_5-1}\to a_{m_5-2}\to a_{m_5-3}\to\dots\to a_{m_2+1}\to a_{m_2+1}\to a_{m_2}$ for the cases (e); $T_4: a_{m_5}\to a_{m_5-1}\to a_{m_5-2}\to a_{m_5-3}\to\dots\to a_{m_1+1}\to a_{m_1+1}\to a_{m_1}$ for the case (f); $T_4: a_m\to a_{m-1}\to a_{m-2}\to a_{m-3}\to\dots\to a_{m_1+1}\to a_{m_1+1}\to a_{m_1}$ for the cases (g) and (h). Observe that the set A_3 and the labels of the vertices of T_4 satisfy the hypothesis of Theorem 2.1. Carry out the transfer T_4 consisting of successive transfers of the first type, keeping $2\alpha_i^{(4)}+1$ elements of A_3 at the vertices a_i of T_4. By Theorem 2.1 the new tree, say G_6, thus formed is graceful. Let A_4 be the set of vertices of A_3 which have been transferred to the last vertex of T_4.

11. Execute the transfer $a_{m_1} \longrightarrow a_{m_1+1}$ (for the cases (f), (g), and (h)), $a_{m_2} \longrightarrow a_{m_2+1}$ (for the cases (c) and (e)), or $a_{m_3} \longrightarrow a_{m_3+1}$ (for the cases (a), (b), (d), and (i)) of the first type and bring back all the elements of A_4 to a_{m_1+1}, a_{m_2+1}, or a_{m_3+1} as the case may be. Obviously, the new tree thus formed, say G_7, is graceful.

12. Now consider the transfer $T_5: \ a_{l+1} \to a_{l+2} \to a_{l+3} \to \ldots \to a_m \to b_1 \to b_2 \to \ldots \to b_n \to k+1$, where $l = m_1$ for the cases (f), (g), and (h), $l = m_2$ for the cases (c) and (e), and $l = m_3$ for the cases (a), (b), (d), and (i). Carry out the transfer T_5 consisting of successive transfers of the first type keeping $2\alpha_i^{(5)} + 1$ elements of A_4 at the vertices a_i and the desired odd number of vertices at $b_j, \ j = 1, 2, \ldots, n$ of T_5. By Theorem 2.1, the new tree, say G_8, thus formed is graceful. LetA_5 be the set of vertex labels of A_4 which have come to the vertex $k+1$ after the transfer T_5.

13. Next consider the transfer $T_6 : k+1 \to q-k-1 \to k+2 \to q-k-2 \to k+3 \to q-k-3 \to \ldots \to p$, where $p = \begin{cases} k + k_1 + 1; & \text{if } m \text{ is odd} \\ q - k - k_1, & \text{if } m \text{ is even} \end{cases}$, $k_1 = s_o + s_e$

Observe that the vertices of transfer T_6 and the elements of A_5 satisfy the hypothesis of Theorem 2.1. Let $s_o = \sum_{i=1}^{m}[o_i]$, $s_e = \sum_{i=1}^{m}[e_i]$, and $s_p = \sum_{i=1}^{m}[p_i]$. Observe that in the transfer T_6, the first s_o vertices are the centers of the odd branches incident on a_is, the next s_e vertices are the centers of the even branches incident on a_is, and the remaining s_p vertices are the pendant vertices incident on a_is. Finally, carry out the transfer T_6 consisting of s_o ($s_o - 1$ if $s_e = 0$) transfers of the first type, followed by s_e transfers of the second type and keep required number of vertices at each vertex of T_6 so that we get the tree $D_6^{(1)}$. By virtue of Theorem 2.1, the tree $D_6^{(1)}$ thus formed after the transfer T_6 has a graceful labeling.

14. Finally attach r pendant vertices to a_0 and assign them the labels $q-r+1$, $q-r+2$, ..., q so that we get the tree D_6. The labeling given to D_6 is obviously graceful.

Case - II: Let $m+n$ be even. Then form a diameter six tree, say G_6 by removing the vertices c_1, c_2, ..., c_r, and b_n from D_6. Let $|E(G_6)| = q_1$. Give a graceful labeling to G_6 by following the steps 1 to 9 while giving a graceful labeling to $D_6^{(1)}$ by replacing $q-r$ with q_1 in the proof for Case - I. Observe that in the graceful labeling of G_6, the vertex a_0 gets the label 0. Now attach the vertices c_1, c_2, ..., c_r, and b_n to a_0 and assign them the labels q_1+1, q_1+2, ..., q_1+r, and q_1+r+1, respectively.

Obviously, the tree $G_6 \cup \{c_1, \ c_2, \ \ldots, c_r, \ b_n\}$ with the labelings mentioned above is graceful with a graceful labeling, say g. Then apply inverse transformation g_{q_1+r+1} to the above labeling of $G_6 \cup \{c_1, \ c_2, \ \ldots, c_r, \ b_n\}$. Now the vertex b_n gets the label 0. Let $deg(b_n) = p$. Finally, attach $p-1$ pendant vertices to b_n and assign them the labels q_1+r+2, q_1+r+3, ..., q_1+r+p, so as to get the tree D_6 with a graceful labeling.
(b) Proof follows on setting $r = 0$ in the proof involving part (a).
(c) Proof follows on setting $n = 0$ in the proof involving part (a).
(d) Proof follows on setting $n = 0$ and $r = 0$ in the proof involving part (a). ■

Example 3.1 The diameter six tree in Fig. 3 is a diameter six of the type (a) in Table 1 in Theorem 3.1(a). Here $q = 114$, $m = 6$, and $n = 3$, a_1 is attached to $(o, 0, 0)$, a_2 is attached to $(o, e, 0)$, a_3 is attached to $(e, o, 0)$, a_4 is attached to (e, o, e), a_5 is attached to $(0, o, e)$, and a_6 is attached to $(0, e, o)$.

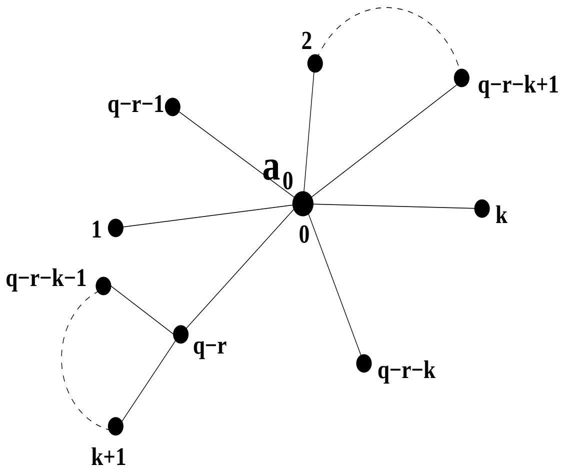

Fig. 2. The graceful tree G

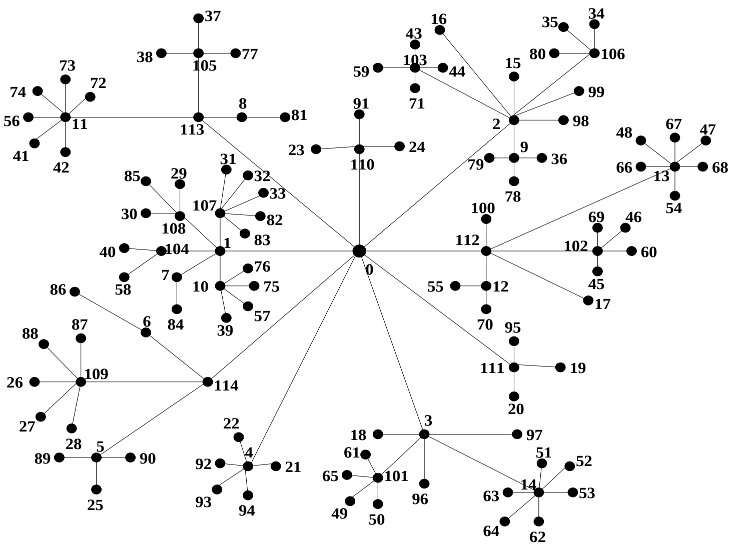

Fig. 3. A diameter six tree of the type (a) in Table 1 in Theorem 3.1 with a graceful labeling

Theorem 3.2. If degrees of a_i and b_j are even, for $i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots, n$, and the centers a_i, $i = 1, 2, \ldots, m$, of diameter four trees are attached to combinations as shown in Table 2 then D_6 given by the following are graceful.

(a): $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$.
(b): $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$.
(c): $D_6 = \{a_0; a_1, a_2, \ldots, a_m; c_1, c_2, \ldots, c_r\}$ with m odd.
(d): $D_6 = \{a_0; a_1, a_2, \ldots, a_m\}$ with m odd.

Proof (a): Case - I Let $m + n$ be odd. Let $|E(D_6)| = q$ and $deg(a_0) = m + n = 2k + 1$. Proceed as per the following steps.

Table 2. Diameter Six Trees of Theorem 3.2

Cases ↓	a_i, $1 \le i \le m_1$	a_i, $m_1+1 \le i \le m_2$	a_i, $m_2+1 \le i \le m_3$	a_i, $m_3+1 \le i \le m_4$	a_i, $m_4+1 \le i \le m_5 = m$
(a)	$(o,0,0)$	$(o,e,0)$	(o,e,e)	(o,o,o)	$(e,0,o)$
(b)	same as (a)	$(o,0,e)$	same as (a)	same as (a)	same as (a)
(c)	same as (a)	same as (b)	(o,o,o)	(e,e,o)	same as (a)
(d)	same as (a)	same as (b)	same as (c)	same as (c)	$(0,e,o)$

1. Repeat Step 1 in the proof involving Case - I of Theorem 3.1.

2. Define integers $\alpha_i^{(j)}$, for $i = 1,2,\ldots,m,\ j = 1,2,3,4,5$ as per the following. **For** $1 \le i \le m_1$**:** $2\alpha_i^{(1)} + 1 = o_i = deg(a_i) - 1$.

For $m_1 + 1 \le i \le m_2$**:** For the case (a): $2\alpha_i^{(1)} + 1 = o_i$, $2[\alpha_i^{(2)} + \alpha_i^{(3)} + 1] = e_i$. For the cases (b), (c), and (d): $2\alpha_i^{(1)} + 1 = o_i$, $2[\alpha_i^{(4)} + \alpha_i^{(5)} + 1] = p_i$.

For $m_2 + 1 \le i \le m_3$**:** For the cases (a) and (b): $2\alpha_i^{(1)} + 1 = o_i$, $2[\alpha_i^{(2)} + \alpha_i^{(3)} + 1] = e_i$, and $2[\alpha_i^{(4)} + \alpha_i^{(5)} + 1] = p_i$. For the cases (c) and (d): $2\alpha_i^{(1)} + 1 = o_i$, $2\alpha_i^{(4)} + 1 = e_i$, $2\alpha_i^{(5)} + 1 = p_i$.

For $m_3 + 1 \le i \le m_4$**:** For the cases (a) and (b): $2\alpha_i^{(1)} + 1 = o_i$, $2\alpha_i^{(2)} + 1 = e_i$, $2\alpha_i^{(5)} + 1 = p_i$. For the cases (c) and (d): $2[\alpha_i^{(1)} + \alpha_i^{(2)} + 1] = o_i$, $2[\alpha_i^{(3)} + \alpha_i^{(4)} + 1] = e_i$, and $2\alpha_i^{(5)} + 1 = p_i$.

For $m_4 + 1 \le i \le m_5 = m$**:** For the cases (a), (be), and (c): $2[\alpha_i^{(1)} + \alpha_i^{(2)} + 1] = o_i$ and $2\alpha_i^{(5)} + 1 = p_i$. For the case (d): $2[\alpha_i^{(3)} + \alpha_i^{(4)} + 1] = e_i$, and $2\alpha_i^{(5)} + 1 = p_i$.

3. Repeat Step 3 in the proof involving Case - I of Theorem 3.1.

4. Define an integer t as $t = b_1$ for the cases (a), (b), and (c) and $t = a_{m_4+1}$ for the case (d). Carry out the transfer $T_1 :\ a_1 \to a_2 \to a_3 \to a_4 \to a_5 \to a_6 \to \ldots \to t$ as in Step 3 of the proof involving Theorem 3.1 and keep $2\alpha_i^{(1)} + 1$ vertices from A at each vertex a_i of T_1. Let A_1 be the set of vertices of A transferred to the vertex t in the transfer T_1.

5. Define an integer t_1 as $t_1 = a_m$ for the cases (a), (b), and (c) and $t_1 = a_{m_4}$ for the case (d). Carry out the transfer $t \longrightarrow t_1$ of the first type and bring back all the elements of A_1 to t_1. Obviously, the new tree thus formed, say G_2, is graceful.

6. Define the transfer T_2 as follows. $T_2 : a_m \to a_{m-1} \to \ldots \to a_{m_1+1} \to a_{m_1}$ for the case (a); $T_2 : a_m \to a_{m-1} \to \ldots \to a_{m_2+1} \to a_{m_2}$ for the case (b); $T_2 : a_m \to a_{m-1} \to \ldots \to a_{m_3+1} \to a_{m_3}$ for the case (c); $T_2 : a_{m_4} \to a_{m_4-1} \to \ldots \to a_{m_3+1} \to a_{m_3}$ for the case (d). Carry out the transfer T_2 keeping $2\alpha_i^{(2)} + 1$ elements of A_1 at the vertices a_i of T_2 as we have done in the proof involving Theorem 3.1. Let A_2 is the set of vertices of A_1 that have come to the vertex a_{m_1}, a_{m_2}, or a_{m_3} as the case may be.

7. Execute the transfer $a_{m_1} \to a_{m_1+1}$ (for the case (a)), $a_{m_2} \to a_{m_2+1}$ (for the case (b)), or $a_{m_3} \to a_{m_3+1}$ (for the cases (c) and (d)) as the case may and bring back all elements of A_2 to a_{m_1+1}, a_{m_2+1}, or a_{m_3+1} as the case may be.

8. Define the transfer T_3 as follows. $T_3 :\ a_{m_1+1} \to a_{m_1+2} \to \ldots \to a_{m_3} \to a_{m_3+1}$ for the case (a); $T_3 :\ a_{m_2+1} \to a_{m_2+2} \to a_{m_2+3} \to \ldots \to a_{m_3} \to a_{m_3+1}$ for the case (b); $T_3 :\ a_{m_3+1} \to a_{m_3+2} \to a_{m_3+3} \to \ldots \to a_{m_4} \to a_{m_4+1}$ for the case (c); $T_3 :\ a_{m_3+1} \to a_{m_3+2} \to a_{m_3+3} \to \ldots \to a_m \to b_1$

for the case (d). Execute the transfer T_3 and keep $2\alpha_i^{(3)}+1$ elements of A_2 at the vertices a_i of T_3 as we have done in the proof involving Theorem 3.1. Let A_3 is the set of vertices of A_2 that have come to the last vertex of T_3.

9. Execute the transfer $a_{m_3+1} \rightarrow a_{m_3}$ (for the cases (a) and (b)) or $a_{m_4+1} \rightarrow a_{m_4}$) (for the case (c)) or $b_1 \longrightarrow a_m$ (for the case (d)) of the first type and bring back all the elements of A_3 to a_{m_3}, a_{m_4}, or a_m as the case may be. Obviously, the new tree thus formed, say G_5, is graceful.

10. Define the transfer T_4 as follows. $T_4 : a_{m_3} \rightarrow a_{m_3-1} \rightarrow \ldots \rightarrow a_{m_2+1} \rightarrow a_{m_2}$ for the case (a); $T_4 : a_{m_3} \rightarrow a_{m_3-1} \rightarrow \ldots \rightarrow a_{m_1+1} \rightarrow a_{m_1}$ for the case (b); $T_4 : a_{m_4} \rightarrow a_{m_4-1} \rightarrow \ldots \rightarrow a_{m_1+1} \rightarrow a_{m_1}$ for the cases (c) and (d). Execute the transfer T_4 consisting of m successive transfers of the first type, keeping $2\alpha_i^{(4)}+1$ elements of A_3 at the vertices a_i of T_4 as we have done in the proof involving Theorem 3.1. Let A_4 be the set of vertices of A_3 which have been transferred to 0.

11. Execute the transfer $a_{m_1} \longrightarrow a_{m_1+1}$ (for the cases (b), (c), and (s)) or $a_{m_2} \longrightarrow a_{m_2+1}$ (for the case (a)) of the first type and bring back all the elements of A_4 to a_{m_1+1} or a_{m_2+1} as the case may be. Obviously, the new tree thus formed, say G_7, is graceful.

12. Now consider the transfer $T_5 :\ a_{r+1} \rightarrow a_{l+2} \rightarrow a_{l+3} \rightarrow \ldots \rightarrow a_m \rightarrow b_1 \rightarrow b_2 \rightarrow \ldots \rightarrow b_n \rightarrow k+1$, where $l =\ m_1$ for the cases (b), (c), and (d) and $l =\ m_2$ for the case (a). Carry out the transfer T_5 consisting of successive transfers of the first type keeping $2\alpha_i^{(5)}+1$ elements of A_4 at the vertices a_i and the desired odd number of vertices at $b_j,\ j\ =\ 1, 2, \ldots, n$ of T_5. By Theorem 2.1(a), the new tree, say G_8, thus formed is graceful. Let A_5 be the set of vertex labels of A_4 which have come to the vertex $k+1$ after the transfer T_5.

13, 14. Repeat Steps 13 and 14 in the proof involving Theorem 3.1 so that one gets back the tree D_6 with a graceful labeling.

Case - II: If $m+n$ is even then the proof follows from that of Case - I by repeating the procedure involving the proof of Theorem 3.1(a) for Case -II. Proofs for the parts (b), (c), and (d) follow from the proof involving the part (a) by setting $r\ =\ 0$; $n\ =\ 0$; and t $n\ =\ 0$ and $r\ =\ 0$. ∎

Example 3.2 The diameter six tree in Fig. 4 is a diameter six of the type (d) in Table 2 in Theorem 3.2(a). Here $q =\ 109$, $m\ =\ 5$, and $n\ =\ 4$, a_1 is attached to $(o, 0, 0)$, a_2 is attached to $(o, 0, e)$, a_3 is attached to (o, o, o), a_4 is attached to (e, e, o), and a_5 is attached to $(0, e, o)$.

Notation 3.1 Let $D_6\ =\ \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ be diameter six tree. We may have one of or both $n\ =\ 0$ and $r\ =\ 0$. For next couple of results we will consistent use the following notations.

n_e = Number of stars adjacent to a_0 with center having odd degree.
n_o = Number of stars adjacent to a_0 with center having even degree, i.e. $n\ =\ n_e + n_o$.

Theorem 3.3. Let $m+n$ be odd, $n_e\ \cong\ 0\ mod\ 4$, degrees of a_i are even, for $i\ =\ 1, 2, 3,\ \ldots,\ m$. If the centers $a_i,\ i\ =\ 1, 2,\ \ldots,\ m$, of diameter four trees are attached to combinations as shown in Tables 1 and 2 then

(a) $D_6\ =\ \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ is graceful.
(b) $D_6 = \{a_0; a_1, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ is graceful.

Proof: **(a)** Consider the part (a) first. Let $|E(D_6)|\ =\ q$ and $deg(a_0)\ =\ m+n\ =\ 2k+1$. Proceed as per the following steps.

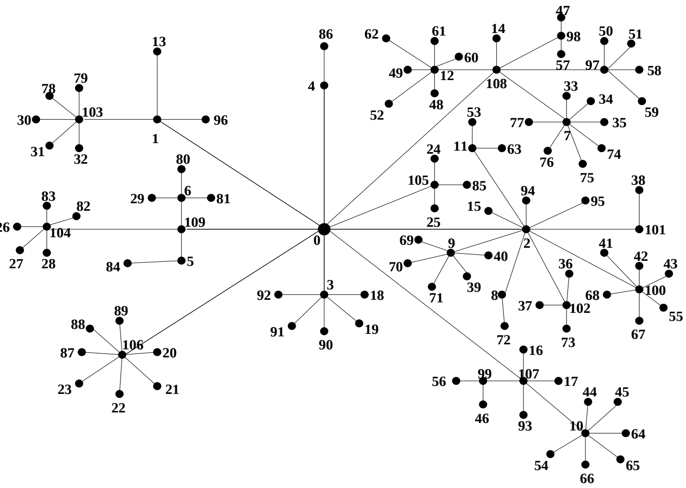

Fig. 4. A diameter six tree of the type (d) in Table 2 in Theorem 3.2 with a graceful labeling.

Repeat Steps 1 to 11 in the proof of Theorem 3.1(a)(or 3.2(a)) for Case -I.

12. Now consider the transfer T_5 : $a_{r+1} \to a_{r+2} \to a_{r+3} \to \ldots \to a_m \to b_1 \to b_2 \to \ldots \to b_n \to k+1$, consisting of $m + n_o$ successive transfers of the first type, followed by $\frac{n_e}{4}$ successive BD8TF from vertex levels in the set A_4. Observe that the transfer T_5 and the set A_4 satisfy the hypothesis of Theorem 2.1.

Carry out the transfer T_5 keeping $2\alpha_i^{(5)} + 1$ elements of A_4 at the vertices a_i, the desired odd number of vertices at b_j, $j = 1, 2, \ldots, n_o$, and the desired even number of vertices at b_j, $j = n_o + 1, n_o + 2, \ldots, n$ of T_5. By Theorem 2.1, the new tree, say G_8, thus formed is graceful. Let A_5 be the set of vertex labels of A_4 which have come to the vertex $k + 1$ after the transfer T_5.

Finally, repeat Steps 13 and 14 in the proof involving Theorem 3.1(a) (or 3.2(a)) for Case -I to get the tree D_6 with a graceful labeling. The proof of part (b) follows on setting $r = 0$ in the proof involving part (a). ■

Theorem 3.4. Let $m + n$ be even, either $n_e \cong 1 \; mod \; 4$ or $n_e \cong 0 \; mod \; 4$ and $n_o \geq 1$, degrees of a_i are even, for $i = 1, 2, 3, \ldots, m$. If the centers a_i, $i = 1, 2, \ldots, m$, of diameter four trees are attached to combinations as shown in Tables 1 and 2 then
(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ is graceful.
(b) $D_6 = \{a_0; a_1, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ is graceful.

Proof : Consider the part (a) first. Designate the vertex b_n as the center of a star adjacent to a_0 with odd (respectively, even) degree if $n_e \cong 1 \; mod \; 4$ (respectively, $n_e \cong 0 \; mod \; 4$, $n_o \geq 1$). Define two integers k_1 and k_2 as

$$k_1 = \begin{cases} n_e - 1 & \text{if } n_e \cong 1 \; mod \; 4 \\ n_e & \text{if } n_e \cong 0 \; mod \; 4 \text{ and } n_o \geq 1 \end{cases} ; k_2 = \begin{cases} n_o & \text{if } n_e \cong 1 \; mod \; 4 \\ n_o - 1 & \text{if } n_e \cong 0 \; mod \; 4 \text{ and } n_o \geq 1 \end{cases}$$

So we have $n = n_o + n_e = k_1 + k_2 + 1$. Form a diameter six tree, say G_6 by removing the vertices $c_1, c_2, \ldots, c_r$, and b_n from D_6. Let $|E(G_6)| = q_1$. Give a graceful labeling to G_6 by following the steps 1 to 12 in the proof of Theorem 3.1(a) (or 3.2(a)) by setting $q - r = q_1$ and replacing n_e

with k_1 and n_o by k_2. Observe that in the graceful labeling of G_6, the vertex a_0 gets the label 0. Now attach the vertices $c_1,\ c_2,\ \ldots,\ c_r$, and b_n to a_0 and assign them the labels $q_1+1,\ q_1+2,\ \ldots,$ q_1+r, and q_1+r+1, respectively. Obviously, the tree $G_6\ \cup\{c_1,\ c_2,\ \ldots, c_r,\ b_n\}$ with the labelings mentioned above is graceful with a graceful labeling, say g. Then apply inverse transformation g_{q_1+r+1} to the above labeling of $G_6\ \cup\{c_1,\ c_2,\ \ldots, c_r,\ b_n\}$. Now the vertex b_n gets the label 0. Let $deg(b_n)\ =\ p$. Finally, attach p pendant vertices to b_n and assign them the labels q_1+r+2, $q_1+r+3,\ \ldots,\ q_1+r+p+1$, so as to get the tree D_6 with a graceful labeling. The proof of part (b) follows if we set $r\ =\ 0$. ■

Theorem 3.5. If is m even, degree of a_i are even, for $i\ =\ 1,2,3,\ldots,m$, the centers $a_i,\ i\ =\ 1,2,\ \ldots,\ m$, of diameter four trees are attached to combinations as shown in Tables 1 and 2 with $m_1\ \geq 3$ and degree of at least one $a_i,\ 1\leq i\leq m_1$ is $\geq\ 4$, then $D_6=\{a_0;a_1,\ldots,a_m\}$ has a graceful labeling.

Proof : Let us designate the vertex a_2 as the center of diameter four tree whose degree upon which at lest 3 odd branches are incident. Let us remove one diameter four tree with center having even degree from D_6. Let us designate this vertex as a_m. Excluding a_0 there are o_m neighbours of a_m. We attach o_m-1 neighbours of a_m to the vertex a_2. Let the resultant tree thus formed be G_6. Obviously it is a diameter six tree of the type $D_6^{(1)}$ in the proof involving Theorem 3.1(a) (or 3.2(a)) for Case - I. Let $|E(G_6)|\ =\ q_1$. Repeat the procedure in the proof of Theorem 3.1(a) (or 3.2(a)) for Case - I by replacing m_1 with m_1-1, m with $m-1$ and $q-r$ with q_1 and give a graceful labeling to G_6.

Observe that the vertex a_2 gets label 1, and the $2\alpha_2^{(1)}+o_m$ neighbours of a_2 get the labels q_1-x, $x+1+i,\ q_1-x-i,\ x\ =\ k+\alpha_1^{(1)}+1,\ i\ =\ 1,2,\ldots,\alpha_2^{(1)}+[\frac{o_m-1}{2}]$. While labeling G_6 allot labels $x+i+2,\ q_1-x-i,\ i\ =\ 1,2,\ldots,[\frac{o_m-1}{2}]$ to o_m-1 neighbours of a_m that were shifted to a_2 while constructing G_6. Next attach the vertex a_m to a_0 and assign label q_1+1. Now move the vertices $x+i+2, q_1-x-i, i\ =\ 1,2,\ldots,[\frac{o_m-1}{2}]$, to a_m. Since $(x+i+2)+(q_1-x-i)\ =\ q_1+2\ =\ 1+(q_1+1)$, for $i\ =\ 1,2,\ldots,[\frac{o_m-1}{2}]$, by Theorem 2.1 the resultant tree, say G_1 thus formed is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+1} to G_1 so that the label of the vertex a_m becomes 0. By Lemma 2.2, g_{q_1+1} is a graceful labeling of G_1. The labelling g_{q_1+1} assigns the label 0 to the vertex a_m. Now attach a new vertex to a_m and assign it the label q_1+2. The resultant tree thus obtained, say G_2 is graceful and the the resultant graceful labeling be g_1. Apply inverse transformation ${g_1}_{q_1+2}$ to G_2 so that the label of the vertex q_1+2 of G_2 becomes 0. By Lemma 2.2, ${g_1}_{q_1+2}$ is a graceful labeling of G_2. Let excluding a_m the number of neighbours of the remaining odd branch be p. Now attach the p pendant vertices adjacent to the vertex labelled 0 and assign them the labels $q_1+3, q_1+4,\ldots,q_1+p+2$. So we get the tree D_6 as desired and the labeling obtained above is a graceful labeling of D_6. ■

Example 3.3 The diameter six tree in Fig. 5 [a] is a diameter six of the type in Theorem 3.5. Here $q=\ 126$, $m\ =\ 8$, each $a_i,\ i\ =\ 1,2,3$, is attached to $(o,0,0)$, a_4 is attached to $(o,0,e)$, each $a_i,\ i\ =\ 5,6,7$, is attached to (o,e,e), a_8 is attached to $(e,0,o)$. We first form the graceful diameter six tree G_6 as in Figure [b] by removing the branch incident on a_8 and three branches incident on it and making two of these branches adjacent to the vertex a_2. Figure [c] represents the tree obtained from the graceful tree in [b] by attaching a vertex to a_0 (with label 0) and assigning the label 122 to it and shifting the branches with labels 7 and 116 from the vertex label 1 to the new vertex (labeled 122). The graceful tree in Figure [d] is obtained by applying inverse transformation to the graceful tree in Figure [c] and attaching a new vertex to the vertex labeled 0 and assigning it the label 123. Finally, the graceful tree D_6 in Figure [e] is obtained by applying inverse transformation to the graceful tree in Figure [d] and attaching three vertices to the vertex labeled 0 and assigning them the labels 124, 125, and 126.

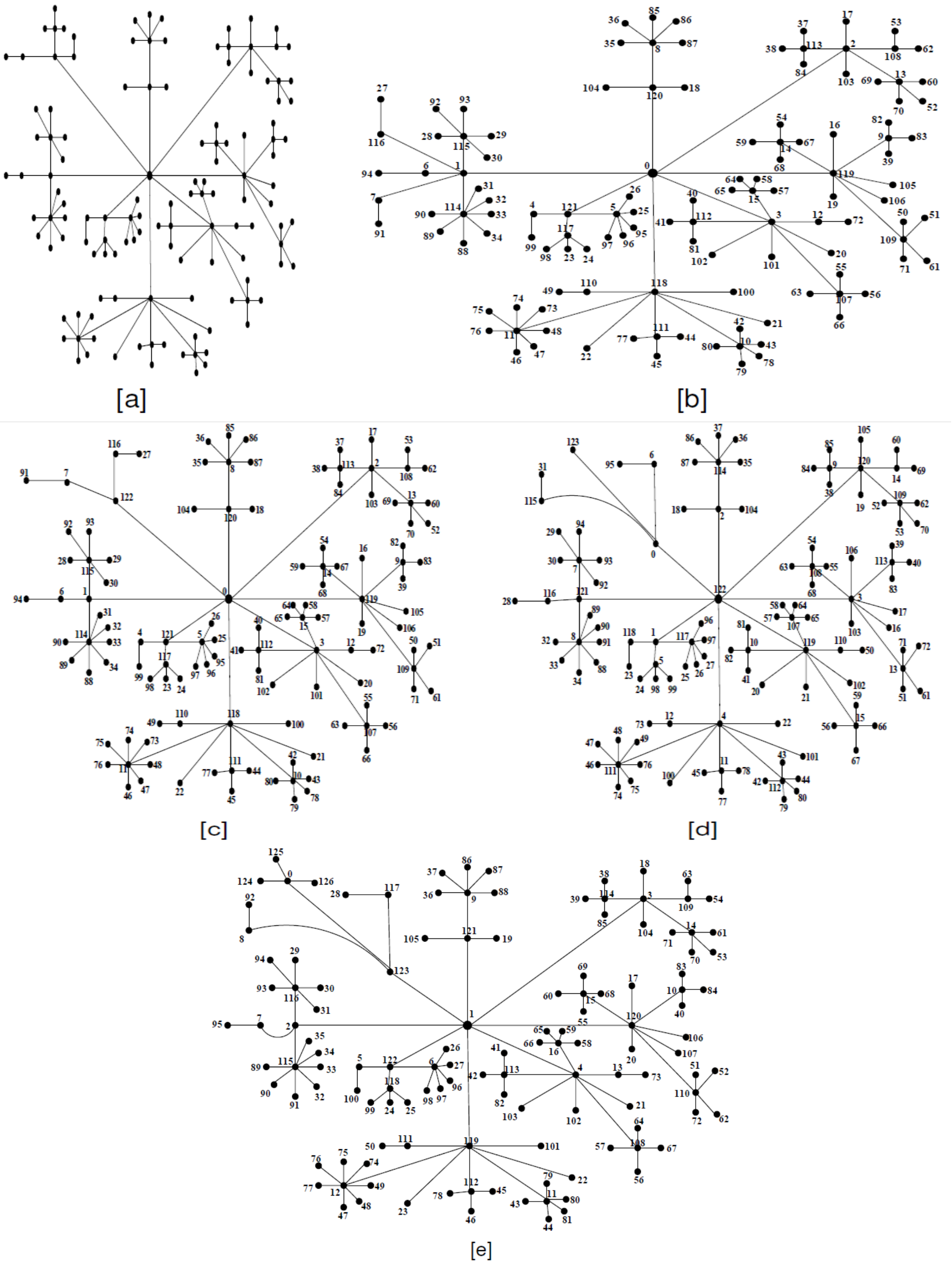

Fig. 5. A diameter six tree of the type in Theorem 3.5 with a graceful labeling.

4 Conclusion

a In this article we have given graceful labelings to some new classes of diameter six trees in which the diameter four trees adjacent to the centers contain six different combinations of odd even and pendant branches. We feel our effort will inspire the researchers of this area to make inroads in the direction of resolving the conjecture of Ringel and Kotzig (1964) which states that all trees are graceful.

b As a sub case of "the graceful tree conjecture" we state the following conjecture.

Conjecture: All trees of diameter six are graceful.

c As a future work from the concepts discussed in this article, one can try out the followings.

i) Giving graceful labelings to some more generalized classes of diameter six trees in which the diameter four trees adjacent to the center may have any combinations of odd, even, and pendant branches.

ii) Giving graceful labelings to some classes of trees with any even diameter.

iii) Giving graceful labelings to all lobsters with diameter six.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Bloom GS, Golomb SW. Applications of numbered undirected graphs. Proc. IEEE. 1977;65:562-570.

[2] Bloom GS, Golomb SW. Numbered complete graphs, unusual rulers, and assorted applications, in Theory and Applications of Graphs. Lecture Notes in Math, Springer-Verlag, New York. 1978;642:53-65.

[3] Ringel G. Problem 25 in Theory of Graphs and Applications. Proceedings of Symposium Smolenice 1963, Prague Publishing House of Czechoslovak Academy of Science. 1963;162.

[4] Kotzig A. On certain vertex valuations of finite graphs. Util. Mathematica. 1973;4:261-290.

[5] Rosa A. On Certain Valuations of the Vertices of a Graph. In Theórie des Graphes, (ed. P. Rosenstiehl), Dunod, Paris. 1968;349-355.

[6] Gallian JA. A dynamic survey of graph labeling. Electronic Journal of Combinatorics, DS6. 2015;18.
Available: http://www.combinatorics.org/Surveys/

[7] Edwards M, Howard L. A survey of graceful trees. Atlantic Electronic Journal of Mathematics. 2006;1:5-30.

[8] Hrnciar P, Havier A. All trees of diameter five are graceful. Discrete Mathematics. 2001;233:133-150.

[9] Robeva E. An extensive survey of graceful trees. Udergraduate Honours Thesis, Standford University, USA; 2011.

[10] Mishra D, Panda AC. A Class of diameter six graceful trees. Journal of Advances in Mathematics (JAM). 2014;9(5):2677-2686.

[11] Mishra D, Panda AC. A Class of Diameter Six Trees with Graceful Labelings. International Journal of Mathematics Trends and Technology (IJMTT). 2014; 9(1):1-11.

[12] Panda AC, Mishra D. Some new classes of graceful diameter six trees. Turkic World Mathematical Society Journal of Applied and Engineering Mathematics. 2015;5(2):269-275.

[13] Bhat-Nayak V, Deshmukh U. New families of graceful banana trees. Proc. Indian Acad. Sci. Math. Sci. 1996;106:201-216.

[14] Chen WC, Lu HI, Lu YN. Operations of interlaced trees and graceful trees. South East Asian Bulletin of Mathematics. 1997;4:337-348.

[15] Jeba Jesintha J. New classes of graceful trees. Ph. D.Thesis, Anna University, Chennai, India; 2005.

[16] Murugan M, Arumugam G. Are banana trees graceful?. Math. Ed. (Siwan). 2001;35:18-20.

[17] Sethuraman G, Jesintha J. All extended banana trees are graceful. Proc. Internat. Conf. Math. Comput. Sci. 2009;1:4-8.

[18] Sethuraman G, Jesintha J. All banana trees are graceful. Advances Appl. Disc. Math. 2009;4:53-64.

[19] Vilfred V, Nicholas T. Banana trees and unions of stars are integral sum graphs. Ars Combin. 2011;102:79-85.

[20] Mishra D, Panda AC. Some new transformations and their applications involving graceful tree labeling. International Journal of Mathematical Sciences and Engineering Aplications. 2013;7(1):239-254.

[21] Mishra D, Panigrahi P. New class of graceful lobsters obtained from diameter four trees. Utilitus Mathematica. 2009;80:183-209.

[22] Mishra D, Panigrahi P. Some graceful lobsters with all three types of branches incident on the vertices of the central path. Computers and Mathematics with Applications. 2008;56:1382-1394.

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